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ON STABILITY OF MOTION RELATIVE TO A PART OF VARIABLES UNDER CONSTANTLY ACTING PERTURBATIONS*

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A number of theorems on stability of motion relative to a part of the variables under constantly acting perturbations are proved with the aid of the method of Liapunov functions. Examples are presented.

1. We consider a system of differential equations of perturbed motion

$$\mathbf{x}^{\cdot}=\mathbf{X}(t,\mathbf{x}),\quad \mathbf{X}(t,0)\equiv 0,\quad \mathbf{x}\in\mathbf{R}^n \tag{1.1}$$

in which /1/ $\mathbf{x}=(y_1,\ldots,y_m,z_1,\ldots,z_p)$, $m>0$, $p\geqslant 0$, $n=m+p$. We assume that: a) the right-hand sides of system (1.1) in the domain

$$t\geqslant 0,\quad \|\mathbf{y}\|\leqslant H>0,\quad \|\mathbf{z}\|<+\infty \tag{1.2}$$

are continuous and satisfy the conditions for the uniqueness of the solution; b) the solutions of system (1.1) are $\mathbf{z}$-continuable. Together with system (1.1) we consider the "perturbed" system

$$\mathbf{x}^{\cdot}=\mathbf{X}(t,\mathbf{x})+\mathbf{R}(t,\mathbf{x}) \tag{1.3}$$

relative to which we assume the fulfillment of conditions a) and b), where, in general, $\mathbf{R}(t,0)\not\equiv 0$. By $\mathbf{x}=\mathbf{x}(t;t_0,\mathbf{x}_0)$ we denote the solution of system (1.3), determined by the initial conditions $\mathbf{x}(t_0;t_0,\mathbf{x}_0)=\mathbf{x}_0$. Generalizing the concepts introduced in /2−6/ to the problem of stability relative to a part of the variables, we make the following definitions.

Definition 1. The motion $\mathbf{x}=0$ of system (1.1) is said to be $\mathbf{y}$-stable under constantly acting perturbations (c.a.p.), small at each instant (small on the average or integrally small)), if for any $\varepsilon>0, t_0\geqslant 0$ (respectively, $\varepsilon>0, t_0\geqslant 0, T>0$ or $\varepsilon>0,\ t_0\geqslant 0$) there exist $\delta_1(\varepsilon,t_0)>0, \delta_2(\varepsilon,t_0)>0$ (respectively, $\delta_1(\varepsilon,t_0,T)>0$, $\delta_2(\varepsilon,t_0,T)>0$ or $\delta_1(\varepsilon,t_0)>0$, $\delta_2(\varepsilon,t_0)>0$) such that every solution $\mathbf{x}(t;t_0,\mathbf{x}_0)$ with $\|\mathbf{x}_0\|<\delta_1$ of any system (1.3) for which there is fulfilled in domain

$$t\geqslant t_0,\quad \|\mathbf{y}\|<\varepsilon,\quad 0\leqslant\|\mathbf{z}\|<+\infty \tag{1.4}$$

the condition

$$\|\mathbf{R}(t,\mathbf{x})\|<\delta_2 \tag{1.5}$$

(respectively,

$$\int_t^{t+T}\sup\,[\|\mathbf{R}(\tau,\mathbf{x})\|:\|\mathbf{y}\|\leqslant\varepsilon, 0\leqslant\|\mathbf{z}\|<+\infty]\,d\tau\leqslant\delta_2\quad\text{for all}\quad t\geqslant t_0 \tag{1.6}$$

or

$$\int_{t_0}^{\infty}\sup\,[\|\mathbf{R}(\tau,\mathbf{x})\|:\|\mathbf{y}\|\leqslant\varepsilon,\ 0\leqslant\|\mathbf{z}\|<+\infty]\,d\tau\leqslant\delta_2 \tag{1.7}$$

satisfies inequality $\|\mathbf{y}(t;t_0,\mathbf{x}_0)\|<\varepsilon$ for all $t\geqslant t_0$.

Definition 2. If in Definition 1 for any $\varepsilon>0$ (for any $\varepsilon>0, T>0$ or for any $\varepsilon>0$) we can choose $\delta_1(\varepsilon)>0, \delta_2(\varepsilon)>0$ respectively, $\delta_1(\varepsilon,T)>0$, $\delta_2(\varepsilon,T)>0$ or $\delta_1(\varepsilon)>0$, $\delta_2(\varepsilon)>0$) not depending on $t_0\geqslant 0$, then the $\mathbf{y}$-stability under c.a.p. small at each instant (small on the average or integrally small) is said to be uniform. (Uniform stability under c.a.p. small at each instant is also called total stability /7,8/).

If in Definition 1 the inequality $\|\mathbf{x}_0\|<\delta_1$ is replaced by the condition $\|\mathbf{y}_0\|<\delta_1$ ($\|\mathbf{z}_0\|<\infty$), then from Definitions 1 and 2 we obtain the definitions of stability (uniform stability) under c.a.p. small at each instant (small on the average or integrally small) of the set

$$\{\mathbf{x}:\ \mathbf{y}=0\} \tag{1.8}$$

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under the assumption that it is invariant relative to system (1.1). It is clear that the $\mathbf{y}$-stability under c.a.p. of motion $\mathbf{x}=0$ of system (1.1) follows from the stability under c.a.p. of invariant set (1.8). It is also obvious that from the stability (uniform stability) under c.a.p. small on the average follows the stability (uniform stability) under c.a.p. small at each instant, and this, in its own turn, implies stability (uniform stability) under integrally small c.a.p.

2. Theorem 1. Assume the existence of a function $V(t, \mathbf{x})$ having continuous and bounded partial derivatives with respect to the coordinates

$$\| \partial V/\partial \mathbf{x} \| \leqslant N = \text{const} \tag{2.1}$$

and satisfying the inequalities

$$V(t, \mathbf{x}) \geqslant a(\| \mathbf{y} \|) \tag{2.2}$$

$$V(t, \mathbf{x}) \leqslant b\left(\left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right), \quad m \leqslant k \leqslant n \tag{2.3}$$

whose time derivative relative to system (1.1)

$$V^{\cdot}_{(1.1)}(t, \mathbf{x}) \leqslant -c\left(\left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right) \tag{2.4}$$

Here $a(r)$, $b(r)$ and $c(r)$ are continuous monotonically increasing functions vanishing when $r = 0$. Then the motion $\mathbf{x} = 0$ of system (1.1) is uniformly $\mathbf{y}$-stable under c.a.p. small at each instant.

Proof. The derivatives of function $V(t, \mathbf{x})$ for systems (1.1) and (1.3) are related by

$$V^{\cdot}_{(1.3)}(t, \mathbf{x}) = V^{\cdot}_{(1.1)}(t, \mathbf{x}) + \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}} \mathbf{R}(t, \mathbf{x}) \tag{2.5}$$

According to (2.3)

$$\left(\sum_{i=1}^{k} x_i^2\right)^{1/2} \geqslant b^{-1}(V(t, \mathbf{x}))$$

(b^{-1} is the function inverse to b) which together with (2.4) leads to the inequality

$$V^{\cdot}_{(1,1)}(t, \mathbf{x}) \leqslant -c(b^{-1}(V, (t, \mathbf{x}))) \tag{2.6}$$

on the basis of (2.6) and (2.1), from (2.5) we obtain

$$V^{\cdot}_{(1.3)}(t, \mathbf{x}) \leqslant -c(b^{-1}(V(t, \mathbf{x}))) + N \| \mathbf{R}(t, \mathbf{x}) \| \tag{2.7}$$

Let $\varepsilon \in (0, H)$ be given. Assume $\delta_1(\varepsilon) = b^{-1}(a(\varepsilon))$, $\delta_2(\varepsilon) = c(b^{-1}(a(\varepsilon))) / N$. If (1.5) is fulfilled in domain (1.4), then

$$V^{\cdot}_{(1.3)}(t, \mathbf{x})\,|_{V(t, \mathbf{x}) = a(\varepsilon)} < 0 \tag{2.8}$$

follows from (2.7).

Consider an arbitrary solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ of system (1.3) with $t_0 \geqslant 0$, $\| \mathbf{x}_0 \| < \delta_1$. By (2.2), $V(t_0, \mathbf{x}_0) < a(\varepsilon)$. Let us show that

$$V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) < a(\varepsilon) \quad \text{for all} \quad t \geqslant t_0 \tag{2.9}$$

We assume, to the contrary, that $V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) < a(\varepsilon)$ when $t \in [t_0, t_1)$, but $V(t_1, \mathbf{x}(t_1; t_0, \mathbf{x}_0)) = a(\varepsilon)$. Then, obviously, $V^{\cdot}_{(1.3)}(t_1, \mathbf{x}(t_1; t_0, \mathbf{x}_0)) \geqslant 0$, which contradicts (2.8). On the basis of (2.2), from (2.9) we conclude that $\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| < \varepsilon$ for all $t \geqslant t_0$. The theorem has been proved.

In particular case, when $k = m$, the following stronger statement is valid.

Theorem 2. Assume the existence of a function $V(t, \mathbf{x})$ satisfying (2.1), (2.2) and

$$V(t, \mathbf{x}) \leqslant b(\| \mathbf{y} \|) \tag{2.10}$$

$$V^{\cdot}(t, \mathbf{x}) \leqslant -c(\| \mathbf{y} \|) \tag{2.11}$$

Then the set (1.8), invariant /1,9/ relative to system (1.1), is uniformly stable under c.a.p. small on the average.

Proof. Since $V(t, 0, \mathbf{z}) \equiv 0$ in accord with (2.10), then from (2.1) follows

$$V(t, \mathbf{x}) \leqslant N \|\mathbf{y}\| \tag{2.12}$$

Let $\varepsilon \in (0, H)$ be given. We introduce the notation (see (1.6))

$$\varphi(t) = \sup [\|\mathbf{R}(t, \mathbf{x})\| : \|\mathbf{y}\| \leqslant \varepsilon,\ 0 \leqslant \|\mathbf{z}\| < \infty] \tag{2.13}$$

and we consider the function /3,4/ $f(t, \mathbf{x}) = V(t, \mathbf{x}) e^{\beta(t)}$. According to (2.1) and (2.13) its derivative relative to system (1.3) satisfies the inequality

$$f^{\cdot}_{(1.3)}(t, \mathbf{x}) = e^{\beta(t)}\beta^{\cdot}(t) V(t, \mathbf{x}) + e^{\beta(t)} V^{\cdot}_{(1.1)}(t, \mathbf{x}) + e^{\beta(t)} \frac{\partial V(t, \mathbf{x})}{\partial \mathbf{x}} \mathbf{R}(t, \mathbf{x}) \leqslant f(t, \mathbf{x}) [\beta^{\cdot}(t) + V^{\cdot}_{(1.1)}(t, \mathbf{x}) / V(t, \mathbf{x}) + N\varphi(t) / V(t, \mathbf{x})] \tag{2.14}$$

when $\|\mathbf{y}\| \leqslant \varepsilon$. We set $\delta_1(\varepsilon) = h\varepsilon / N$, where $h = h(\varepsilon) \in (0, 1)$ will be chosen later. By (2.12)

$$V(t, \mathbf{x}) \leqslant h\varepsilon \quad \text{for } \|\mathbf{y}\| \leqslant \delta_1 \tag{2.15}$$

In the domain

$$t \geqslant 0, \quad \delta_1 \leqslant \|\mathbf{y}\| \leqslant \varepsilon, \quad 0 \leqslant \|\mathbf{z}\| < \infty \tag{2.16}$$

there hold the inequalities

$$a(\delta_1) \leqslant V(t, \mathbf{x}) \leqslant N\varepsilon, \quad V^{\cdot}_{(1.1)}(t, \mathbf{x}) \leqslant -c(\delta_1) \tag{2.17}$$

consequently, $f_{(1.3)}$ satisfies the estimate

$$f_{(1.3)}(t, \mathbf{x}) \leqslant f(t, \mathbf{x}) [\beta^{\cdot}(t) - c(\delta_1) / (N\varepsilon) + N\varphi(t) / a(\delta_1)] \tag{2.18}$$

in (2.16). In (1.6) we choose $\delta_2(\varepsilon, T)$ from the condition

$$\delta_2(\varepsilon, T) = (1 - q) c(\delta_1) a(\delta_1) T / (N^2\varepsilon) \tag{2.19}$$

where $q = q(\varepsilon) \in (0,1)$ will be determined later, and we construct the function $\psi(t)$ $(t \geqslant 0)$ such that the equality

$$\int_{\mu T}^{(\mu+1)T} \psi(t)\, dt = \int_{\mu T}^{(\mu+1)T} [(1 - q) c(\delta_1)/(N\varepsilon) - N\varphi(t)/a(\delta_1)]\, dt \tag{2.20}$$

is fulfilled for all $\mu = 0, 1, 2, \ldots$. On the strength of (1.6) and (2.19) we can take it that $\psi(t) \geqslant 0$ for all $t \geqslant 0$.

We set

$$\beta(t) = \int_0^t [-\psi(\tau) + (1 - q) c(\delta_1)/(N\varepsilon) - N\varphi(\tau)/a(\delta_1)]\, d\tau \tag{2.21}$$

From (2.18) and (2.21) it follows that

$$f_{(1.3)}(t, \mathbf{x}) \leqslant f(t, \mathbf{x}) [-\psi(t) - qc(\delta_1)/(N\varepsilon)] < 0 \tag{2.22}$$

in domain (2.16). According to (2.20), $\beta(\mu T) = 0$; consequently,

$$\int_0^t \psi(\tau)\, d\tau \leqslant \int_0^T \psi(\tau)\, d\tau = \int_0^T ((1 - q) c(\delta_1)/(N\varepsilon) - N\varphi(\tau)/a(\delta_1))\, d\tau \leqslant (1 - q) c(\delta_1) T/(N\varepsilon), \quad \int_0^t (N\varphi(\tau)/a(\delta_1))\, d\tau \leqslant (1 - q) c(\delta_1) T/(N\varepsilon)$$

for any $t \in [0, T]$ and, therefore,

$$|\beta(t)| \leqslant \Delta \equiv 3(1 - q) c(\delta_1) T/(N\varepsilon) \quad \text{for all } t \geqslant 0 \tag{2.23}$$

From (2.15), (2.23) and (2.2) we conclude that

$$f(t, \mathbf{x})|_{\|\mathbf{y}\| \leqslant \delta_1} \leqslant h\varepsilon e^{\Delta}, \quad f(t, \mathbf{x})|_{\|\mathbf{y}\| = \varepsilon} \geqslant a(\varepsilon) e^{-\Delta}$$

We now choose numbers $h(\varepsilon)$ and $q(\varepsilon)$ from the conditions

$$h(\varepsilon)\,\varepsilon e < a(\varepsilon)/e, \quad \Delta \equiv 3(1-q(\varepsilon))\,c(\delta_1(\varepsilon))T/(N\varepsilon) < 1 \tag{2.24}$$

Then, obviously,

$$\sup[f(t,\mathbf{x}): \|\mathbf{y}\| \leqslant \delta_1] \leqslant h\varepsilon e < a(\varepsilon)/e \leqslant \inf[f(t,\mathbf{x}): \|\mathbf{y}\| = \varepsilon] \tag{2.25}$$

Since (2.22) is fulfilled in (2.16), from (2.25) we conclude that inequality $\|\mathbf{y}(t;\,t_0,\mathbf{x}_0)\| < \varepsilon$ is valid for the solution $\mathbf{x}(t;\,t_0,\mathbf{x}_0)$ of system (1.3) with $t_0 \geqslant 0$, $\|\mathbf{y}_0\| < \delta_1$, $\|\mathbf{z}_0\| < \infty$ for all $t \geqslant t_0$, since otherwise we would find two instants t_1 and t_2 such that

$$\|\mathbf{y}(t_1;\,t_0,\mathbf{x}_0)\| = \delta_1, \ \|\mathbf{y}(t_2;\,t_0,\mathbf{x}_0)\| = \varepsilon, \ \delta_1 < \|\mathbf{y}(\tau;\,t_0,\mathbf{x}_0)\| < \varepsilon \quad \text{for all} \quad \tau \in (t_1, t_2),$$

and, therefore,

$$a(\varepsilon)/e \leqslant f(t_2, \mathbf{x}(t_2;\,t_0,\mathbf{x}_0)) = f(t_1, \mathbf{x}(t_1;\,t_0,\mathbf{x}_0)) + \int_{t_1}^{t_2} f^{\cdot}_{(1.3)}(\tau, \mathbf{x}(\tau;\,t_0,\mathbf{x}_0))\,d\tau < f(t_1, \mathbf{x}(t_1;\,t_0,\mathbf{x}_0)) \leqslant h\varepsilon e$$

which contradicts the first of inequalities (2.24). The theorem has been proved.

Note. Theorems 1 and 2 generalize the results of /2–4/ to the problem of stability relative to a part of the variables; in addition, Theorem 2 strengthens Karimov's theorem /10/.

Corollary. If the functions $\mathbf{X}$ and $\partial\mathbf{X}/\partial\mathbf{x}$ are continuous and bounded in domain (1.2), while the invariant set (1.8) is uniformly asymptotically stable, then it is uniformly stable under c.a.p. small on the average.

Indeed, under the assumptions made, as was shown in /9/ (see /1/ as well), a function $V(t,\mathbf{x})$ satisfying the hypotheses of Theorem 2 exists.

Theorem 3. Assume that for any $T > 0$ there exists $L(T) > 0$ such that condition $\|\mathbf{X}(t,\mathbf{x}') - \mathbf{X}(t,\mathbf{x}'')\| \leqslant L\|\mathbf{x}' - \mathbf{x}''\|$ is fulfilled in domain $0 \leqslant t \leqslant T$, $\|\mathbf{x}\| \leqslant H$. If functions $V(t,\mathbf{x})$ and $W(t,\mathbf{x})$, exist, satisfying inequalities (2.1) and

$$a(\|\mathbf{y}\|) \leqslant V(t,\mathbf{x}) \leqslant b(\|\mathbf{y}\|), \quad W(t,\mathbf{x}) \geqslant c(\|\mathbf{y}\|) \tag{2.26}$$

in domain (1.2), and, in addition, if the condition

$$V^{\cdot}_{(1.1)}(t,\mathbf{x}) + W(t,\mathbf{x})_{\lambda \leqslant \|\mathbf{y}\| \leqslant \mu,\ 0 \leqslant \|\mathbf{z}\| < \infty} \rightrightarrows 0 \quad \text{as} \quad t \to \infty$$

is fulfilled for any λ and μ such that $0 < \lambda < \mu < H$, then the motion $\mathbf{x} = 0$ of system (1.1) is uniformly $\mathbf{y}$-stable under c.a.p. small at each instant.

The proof is obtained by a slight modification of that of Theorem 1, allowing for the results in /11/.

Theorem 4. Assume the existence of a function $V(t,\mathbf{x})$ satisfying conditions (2.1) and (2.2), such that

$$V^{\cdot}_{(1.1)}(t,\mathbf{x}) \leqslant 0 \tag{2.27}$$

Then the motion $\mathbf{x} = 0$ is uniformly $\mathbf{y}$-stable under integrally small c.a.p. If moreover $V(t,\mathbf{x})$ satisfy the inequality (2.10) then the invariant set (1.8) is uniformly stable under integrally small c.a.p.

Proof. From (2.1) it follows that

$$V(t,\mathbf{x}) \leqslant N\|\mathbf{x}\| \tag{2.28}$$

Let $\varepsilon \in (0, H)$ be given. Set $\delta_1(\varepsilon) = \delta_2(\varepsilon) = {}^1/_2\,a(\varepsilon)/N$. For the solution $\mathbf{x}(t;\,t_0,\mathbf{x}_0)$ of system (1.3) with $t_0 \geqslant 0$, $\|\mathbf{x}_0\| < \delta_1$, by virtue of (2.2), (2.5), (2.1), (2.27), (2.28) and (1.7), we have

$$a(\|\mathbf{y}(t;\,t_0,\mathbf{x}_0)\|) \leqslant V(t, \mathbf{x}(t;\,t_0,\mathbf{x}_0)) = V(t_0,\mathbf{x}_0) + \int_{t_0}^{t} V^{\cdot}_{(1.3)}(\tau, \mathbf{x}(\tau;\,t_0,\mathbf{x}_0))\,d\tau \leqslant N\|\mathbf{x}_0\| +$$
$$\int_{t_0}^{t} V^{\cdot}_{(1.1)}(\tau, \mathbf{x}(\tau;\,t_0,\mathbf{x}_0))\,d\tau + N\int_{t_0}^{\infty} \|\mathbf{R}(\tau, \mathbf{x}(\tau;\,t_0,\mathbf{x}_0))\|\,d\tau < {}^1/_2 a(\varepsilon) + {}^1/_2 a(\varepsilon) = a(\varepsilon)$$

Consequently, $\|\mathbf{y}(t;\,t_0,\mathbf{x}_0)\| < \varepsilon$ for all $t \geqslant t_0$. The theorem's second assertion is proved an-analogously with the trivial replacement of (2.28) by (2.12).

Notes. 1°. The first assertion of Theorem 4 generalizes the results of /5/ to the problem of stability relative to a part of the variables.

2°. In Theroems 1–4 we can waive the smoothness of function V, having replaced condition (2.1) by the weaker $|V(t, \mathbf{x}') - V(t, \mathbf{x}'')| \leqslant N \|\mathbf{x}' - \mathbf{x}''\|$; in this connection, by $V^{\cdot}$ we should understand the generalized derivative (see /12,13/, for instance).

3. **Example 1.** Let us consider the equations of motion of a holonomic mechanical system in Lagrange coordinates

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} \quad (i = 1, \ldots, n) \tag{3.1}$$

Here $T = T_2 + T_1 + T_0$ is the kinetic energy (T_s is an s th-degree form in $q_1^{\cdot}, \ldots, q_n^{\cdot}$) and $U(\mathbf{q})$ is the potential energy. Assume that system (3.1) has a particular solution (the equilibrium position)

$$\mathbf{q} = \mathbf{q}^{\cdot} = 0 \tag{3.2}$$

If T does not depend explicitly on time, then Eqs.(3.1) admit of a (generalized) energy integral

$$H \equiv T_2 - T_0 + U = \text{const} \quad \left(2T_2 = \sum_{i,j=1}^{n} a_{ij}(\mathbf{q}) q_i^{\cdot} q_j^{\cdot}\right) \tag{3.3}$$

The derivative $H^{\cdot}$ relative to the "perturbed" system

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} + R_i \quad (i = 1, \ldots, n) \tag{3.4}$$

has the form

$$H^{\cdot}_{(3.4)} = \sum_{i=1}^{n} R_i q_i^{\cdot} \tag{3.5}$$

If T_2 is positive definite with respect to $q_1^{\cdot}, \ldots, q_n^{\cdot}$ and $U - T_0$ is positive definite with respect to $q_1, \ldots, q_m$, then the equilibrium position (3.2) is uniformly stable relative to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}$ under integrally small c.a.p. R_i. If, in addition, the constraints imposed on the system are independent of time ($T \equiv T_2$, $T_0 \equiv 0$), U admits of an infinitesimal upper bound with respect to $q_1, \ldots, q_m$, and the coefficients $a_{ij}(\mathbf{q})$ are bounded, then the set $\{(\mathbf{q}, \mathbf{q}^{\cdot}): q_1 = \ldots = q_m = q_1^{\cdot} = \ldots = q_n^{\cdot} = 0\}$ invariant relative to system (3.1) is uniformly stable under integrally small c.a.p. R_i.

Example 2. The motion of a holonomic mechanical system with time-independent constraints, under gyroscopic and, perhaphs, dissipative forces, is described by the system of Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = \sum_{j=1}^{n} g_{ij} q_j^{\cdot} - \frac{\partial f}{\partial q_i^{\cdot}} \quad (i = 1, \ldots, n;\ g_{ij} = -g_{ji}) \tag{3.6}$$

Relative to system (3.6), $T^{\cdot}_{(3.6)} = -2f$, while the derivative of the sume function T relative to the perturbed system

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = \sum_{j=1}^{n} g_{ij} q_j^{\cdot} - \frac{\partial f}{\partial q_i^{\cdot}} + R_i \quad (i = 1, \ldots, n) \tag{3.7}$$

has the form

$$T^{\cdot}_{(3.7)} = -2f + \sum_{i=1}^{n} R_i q_i^{\cdot} \tag{3.8}$$

If $f \geqslant 0$, T is positive definite with respect to $q_1^{\cdot}, \ldots, q_n^{\cdot}$, and the coefficients $a_{ij}(\mathbf{q})$ (see (3.3)) are bounded, then the set

$$\{(\mathbf{q}, \mathbf{q}^{\cdot}): \mathbf{q}^{\cdot} = 0\} \tag{3.9}$$

invariant relative to system (3.6) is uniformly stable under integrally small c.a.p.. If, additionally, f is positive definite with respect to $q_1^{\cdot}, \ldots, q_n^{\cdot}$, then the invariant set (3.9) is uniformly stable under c.a.p. small on the average.

4. The comparison principle with a vector-valued Liapunov function /6,14/ in Khatvani's form /15/ can be extended to the problem of y-stability under c.a.p.

Theorem 5. Assume that:

I. A vector-valued function $\mathbf{V}(t, \mathbf{x}) = (V_1, (t, \mathbf{x}), \ldots, V_k(t, \mathbf{x}))$ exists, satisfying the following conditions:

1) $\mathbf{V}(t, \mathbf{x})$ and $\mathbf{V}^{\cdot}_{(1.1)}(t, \mathbf{x})$ are continuous, $\mathbf{V}(t, 0) \equiv \mathbf{V}^{\cdot}_{(1.1)}(t, 0) \equiv 0$;

2) for some l, $1 \leqslant l \leqslant k$, $V_1 \geqslant 0, \ldots, V_l \geqslant 0$, while

$$V_1(t_1, \mathbf{x}) + \ldots + V_l(t, \mathbf{x}) \geqslant a(\|\mathbf{y}\|) \tag{4.1}$$

3) $\|\mathbf{V}(t, \mathbf{x}') - \mathbf{V}(t, \mathbf{x}'')\| \leqslant N \|\mathbf{x}' - \mathbf{x}''\|$, $N = \mathrm{const}$;

4) $V^{\cdot}_{(1.1)}$ satisfies the system of differential inequalities

$$\mathbf{V}^{\cdot}_{(1.1)}(t, \mathbf{x}) \leqslant \mathbf{f}(t, \mathbf{x}, \mathbf{V}(t, \mathbf{x})) \tag{4.2}$$

II. 1) A vector-valued function $\mathbf{f}(t, \mathbf{x}, \mathbf{V})$ is defined and is continuous in the domain

$$t \geqslant 0, \quad \|\mathbf{y}\| \leqslant H, \quad \|\mathbf{z}\| < +\infty, \quad \|\mathbf{V}\| < A$$

where $A = \infty$ or $A > \sup [\|\mathbf{V}(t, \mathbf{x})\| : t \geqslant 0, \|\mathbf{y}\| \leqslant H]$;

2) each of the functions $f_s(t, \mathbf{x}, \mathbf{V})$ is nondecreasing with respect to $V_1, \ldots, V_{s-1}, V_{s+1}, \ldots, V_k$;

3) $\mathbf{f}(t, 0, 0) \equiv 0$.

Denote $\alpha = (\omega_1, \ldots, \omega_l)$ and consider the auxiliary system

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}), \quad \boldsymbol{\omega}^{\cdot} = \mathbf{f}(t, \mathbf{x}, \boldsymbol{\omega}) \tag{4.3}$$

If under the condition $\omega_{10} \geqslant 0, \ldots, \omega_{l0} \geqslant 0$ the solution $(\mathbf{x} = 0, \boldsymbol{\omega} = 0)$ of system (4.3) is α-stable (uniformly α-stable) under c.a.p. small at each instant, small on the average or integrally small, then the motion $\mathbf{x} = 0$ of system (1.1) is $\mathbf{y}$-stable (uniformly $\mathbf{y}$-stable) under c.a.p. small at each instant, small on the average or integrally small, respectively.

Proof. According to condition I—3) and to (4.2) the generalized derivative $\mathbf{V}^{\cdot}_{(1.3)}$ satisfies the inequality

$$\mathbf{V}^{\cdot}_{(1.3)}(t, \mathbf{x}) \leqslant \mathbf{f}(t, \mathbf{x}, \mathbf{V}(t, \mathbf{x})) + N \|\mathbf{R}(t, \mathbf{x})\| \mathbf{b}, \quad \mathbf{b} = (1, \ldots, 1)$$

Together with (4.3) we consider a second auxiliary system

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}) + \mathbf{R}(t, \mathbf{x}), \quad \boldsymbol{\omega}^{\cdot} = \mathbf{f}(t, \mathbf{x}, \boldsymbol{\omega}) + N \|\mathbf{R}(t, \mathbf{x})\| \mathbf{b} \tag{4.4}$$

We carry the proof out for c.a.p. small at each instant (the proof is completely analogous for c.a.p. small on the average or integrally small).

The zero solution of system (4.3) is α-stable under c.a.p. small at each instant; therefore, for any $\varepsilon \in (0, H)$, $t_0 \geqslant 0$ there exist $\eta_1(\varepsilon, t_0) > 0$ and $\eta_2(\varepsilon, t_0) > 0$ such that every solution $(\mathbf{x}(t; t_0, \mathbf{x}_0), \boldsymbol{\omega}(t; t_0, \mathbf{x}_0, \boldsymbol{\omega}_0))$ with $\|\mathbf{x}_0\| < \eta_1, \|\boldsymbol{\omega}_0\| < \eta_1$ of any system (4.4) for which there is fulfilled in domain

$$t \geqslant t_0, \sum_{s=1}^{l} |\omega_s| \leqslant a(\varepsilon)$$

the condition

$$\|\mathbf{R}(t, \mathbf{x})\| < \eta_2, \quad N \|\mathbf{R}(t, \mathbf{x})\| \|\mathbf{b}\| < \eta_2 \tag{4.5}$$

satisfies for all $t \geqslant t_0$ the inequality

$$\sum_{s=1}^{l} |\omega_s(t; t_0, \mathbf{x}_0, \boldsymbol{\omega}_0)| < a(\varepsilon) \tag{4.6}$$

For the numbers $\eta_1(\varepsilon, t_0)$ and $\eta_2(\varepsilon, t_0)$ there exists $\delta_1(\varepsilon, t_0)$ and $\delta_2(\varepsilon, t_0)$, $0 < \delta_1 < \eta_1$, $0 < \delta_2 < \eta_2$, such that $\|\mathbf{V}(t_0, \mathbf{x}_0)\| < \eta_1$ when $\|\mathbf{x}_0\| < \delta_1$, and (4.5) follows from (1.5).

Let $\mathbf{x}(t; t_0, \mathbf{x}_0)$ be a solution of system (1.3) with $\|\mathbf{x}_0\| < \delta_1$. With it we associate the upper solution /16/ $\boldsymbol{\omega}^+(t; t_0, \mathbf{x}_0, \boldsymbol{\omega}_0)$ of problem

$$\begin{aligned} &\boldsymbol{\omega}^{\cdot} = \mathbf{f}(t, \mathbf{x}(t; t_0, \mathbf{x}_0), \boldsymbol{\omega}) + N \|\mathbf{R}(t, \mathbf{x}(t; t_0, \mathbf{x}_0))\| \mathbf{b} \\ &\boldsymbol{\omega}(t_0) \equiv \boldsymbol{\omega}_0 = \mathbf{V}(t_0, \mathbf{x}_0) \end{aligned} \tag{4.7}$$

By the choice of δ_1 we have $\|\omega_0\| < \eta_1$ and, consequently, inequality (4.6) is valid. On the strength of II—2) the right-hand sides of system (4.7) satisfy the Wazhewski condition /16/, and since

$$\mathbf{V}^{\boldsymbol{\cdot}}_{(1.3)}(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant \mathbf{f}(t, \mathbf{x}(t; t_0, \mathbf{x}_0), \mathbf{V}(t, \mathbf{x}(t; t_0, \mathbf{x}_0))) + N \| \mathbf{R}(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \| \mathbf{b}$$

we conclude, on the basis of /16/, that

$$\mathbf{V}(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant \omega^+(t; t_0, \mathbf{x}_0, \omega_0) \tag{4.8}$$

From (4.1), (4.8) and (4.6) follows

$$a(\| \mathbf{y}(t; t_0, \mathbf{x}_0) \|) \leqslant \sum_{s=1}^{l} V_s(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant \sum_{s=1}^{l} \omega_s^+(t; t_0, \mathbf{x}_0, \omega_0) < a(\varepsilon)$$

whence $\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| < \varepsilon$ for all $t \geqslant t_0$. If the α-stability of the zero solution of system (4.3) under c.a.p. small at each instant is uniform, then the numbers η_1, η_2, δ_1 and δ_2 are independent of t_0. The theorem is proved.

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